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ON SOLVING PURSUIT GAME PROBLEMS

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Two methods are proposed for successive approximations of the value function of a pursuit game with limited time and with payoff function $\min_{\tau\in[0,t]} H(x(\tau), y(\tau))$, which are used directly for the constructive design of successive pursuit and evasion strategies which permit ε-optimal strategies to be found in any $\varepsilon > 0$. Necessary and sufficient conditions for some function to be the value function of the pursuit game being analyzed are derived as well. References [1-3] were devoted to sequential methods of constructing the value function or the payoff minimax in game problems of encounter at a specified instant. These methods were used in [1, 2] to construct maximal stable bridges the strategies extremal to which solve the corresponding problem, as is well known from [4]. Sequential procedures for constructing maximal stable bridges without a preliminary construction of the value function or the payoff minimax also were examined in [1, 2] and in [5].

1. Let the motions of a pursuer P and an evader E be described by the equations

$$x^{\cdot} = f(x, u), \quad x \in R^n, \quad u \in K_P \tag{1.1}$$
$$y^{\cdot} = g(y, v), \quad y \in R^m, \quad v \in K_E \tag{1.2}$$

where $K_P \subset R^k$ and $K_E \subset R^l$ are convex compacta. We make the following assumptions concerning system (1.1) ((1.2)):

a) the function $f(x, u)$ $(g(y, v))$ is defined and continuous on $R^n \times K_P$ $(R^m \times K_E)$ and satisfies a local Lipschitz condition in x (y) with a constant not depending on u (v);

b) $\| f(x, u) \| \leqslant \lambda (1 + \| x \|)$ $(\| g(y, v) \| \leqslant \lambda (1 + \| y \|))$, $\lambda > 0$ for all $(x, u) \in R^n \times K_P$ $((y, v) \in R^m \times K_E)$;

c) the measurable program controls $u(\cdot)$ $(v(\cdot))$ with values from K_P (K_E) are the admissable controls;

d) the set $\{f(x, u) \mid u \in K_P\}$ $(\{g(y, v) \mid v \in K_E\})$ is convex for any $x \in R^n$ $(y \in R^m)$.

Further, we assume as specified the initial state $(x_0, y_0, T) \in R^n \times R^m \times$

$(R_+^1 \setminus \{0\})$ of the game, where $x_0 \in R^n$ and $y_0 \in R^m$ are the initial positions of players P and E, while $T \in R_+^1 \setminus \{0\}$ $(R_+^1 = \{t \in R^1 \mid t \geqslant 0\})$ is the game's duration.

We define the class of strategies of player P (E) as follows. A strategy U^q (V^r) is an ordered collection of mappings $U^q = (a^q, \ldots, a^\circ)$ $(V^r = (b^r, \ldots, b^\circ))$ given on $R^n \times R^m \times (R_+^1 \setminus \{0\})$. The mapping a^i (b^i) for $i \in [0 : q]$ $(i \in [0 : r])$ associates with the state $(x, y, t) \in R^n \times R^m \times (R_+^1 \setminus \{0\})$ an admissable control $u_i(\cdot)$ $(v_i(\cdot))$ together with its interval $[0, t_i >, t_i \in (0, t]$, where for $i \in [1 : q]$ $(i \in [1 : r])$ either $[0, t_i > = [0, t_i), t_i \in (0, t)$, or $[0, t_i > = [0, t]$, while $[0, t_i > = [0, t]$ when $i = 0$.

Let us explain how the strategy U^q (V^r) is realized in the game from the initial state $(x^\circ, y^\circ, t^\circ) \in R^n \times R^m \times (R_+^1 \setminus \{0\})$. The mapping a^q (b^r) associates with the initial state an admissible control $u_q(\tau)$, $\tau \in [0, t_q >$ $(v_r(\tau)$, $\tau \in [0, t_r >)$ which determines the solution of Eq. (1.1) ((1.2)) on the appropriate interval, with the initial condition $x(0) = x^\circ$ $(y(0) = y^\circ)$. As a consequence of the assumptions a)-c) made this solution exists and is unique on the whole inteval $[0, t_q]$ $([0, t_r])$. Then, when $q = 0$ $(r = 0)$ or when $[0, t_q > = [0, t^\circ]$ $([0, t_r > = [0, t^\circ])$ by a realization of strategy U^q (V^r) we mean the corresponding solution of Eqs. (1.1) ((1.2)). However, when $q > 0$ $(r > 0)$ and

$$[0, t_q > = [0, t_q),\ t_q \in (0, t^\circ)\ ([0, t_r > = [0, t_r),\ t_r \in (0, t^\circ))$$

we consider the state (x_q^1, y_q^1, t_q^1) $((x_r^1, y_r^1, t_r^1))$ which is realized by the instant t_q (t_r). In this state now the mapping a^{q-1} (b^{r-1}) determines on the corresponding interval the solution of Eq. (1.1) ((1.2)) with initial condition $x(0) = x_q^1$ $(y(0) = y_r^1)$ by means of the admissible control $u_{q-1}(\cdot)$ $(v_{r-1}(\cdot))$. Continuing further, similar arguments convince us that any strategy U (V) in the game from the initial state $(x^\circ, y^\circ, t^\circ)$ generates, in general, jointly with some strategy V (U) the unique solution of Eqs. (1.1) and ((1.2)): $x(t) = x(t, x^\circ \mid U, V)$ $(y(t) = y(t, y^\circ \mid U, V))$ now defined on the whole interval $[0, t^\circ]$, with the initial condition $x(0) = x^\circ$ $(y(0) = y^\circ)$. (Here we admit that in the formation of this solution some of the mappings a^i (b^i), $i \in [0 : s]$, $s < q$ $(s < r)$ may be unessential.)

The payoff function in the game from the initial state (x_0, y_0, T) in the situation (U, V) is defined by the equality

$$K(x_0, y_0, T \mid U, V) = \min_{t \in [0, T]} H(x(t), y(t)) \tag{1.3}$$

$x(t) = x(t, x_0 \mid U, V)$ and $y(t) = y(t, y_0 \mid U, V)$, where the function $H(x, y)$ is defined and continuous on $R^n \times R^m$. We assume that player E maximizes and player P minimizes (1.3). We note as well that we are examining a game with complete information, i.e., each player knows the opponent's dynamics and the game's current state. In addition, if necessary we assume that the players know the entire previous history of the game.

To solve the game from the initial state (x_0, y_0, T) we imbed it into a set of states D_δ, where

$$D_\delta = \bigcup_{(x, y, t) \in S_\delta(x_0, y_0, T)} \bigcup_{t' \in [0, t]} C^{t-t'}(x) \times C^{t-t'}(y) \times \{t'\}$$

$S_\delta(x_0, y_0, T) \subset R^n \times R^m \times R_+^1$ is a closed sphere of radius $\delta \in (0, T]$ with its center at (x_0, y_0, T) and $C^{t-t'}(x)$ $(C^{t-t'}(y))$ is the attainability set of system (1.1) ((1.2)) from the initial position $x(0) = x$ $(y(0) = y)$ by the instant $t - t'$. With the assumptions a)-d) made the sets $C^t(x)$ $(C^t(y))$ for any $x \in R^n$ $(y \in R^m)$ and $t \in R_+^1$ and, consequently, the set D_δ, are compact [6]. We introduce the notation $D_\delta^\circ = \{(x, y, t) \in D_\delta \mid t > 0\}$.

2. Definition 1. Let a function $w(\cdot)$ be defined on D_δ. We say that function $w(\cdot)$ belongs to a set $W_-(D_\delta)$ if:

1) $w(x, y, 0) = H(x, y)$ for any point $(x, y, 0) \in D_\delta$;

2) $w(\cdot) \in C(D_\delta)$ ($C(D_\delta)$ is a set of functions continuous on D_δ

3) a strategy V' of the player E exists such that the inequalities $K(x, y, t \mid U, V') \geqslant w(x, y, t)$ and $w(x(\tau, x \mid U, V'), y(\tau, y \mid U, V') t - \tau) \geqslant w(x, y, t)$ for all $\tau \in [0, \tau']$ with some τ', $\tau' \in (0, t]$ are valid at each point $(x, y, t) \in D_\delta$ for any strategy U of player P.

Lemma 1. Set $W_-(D_\delta)$ is nonempty.

To prove the lemma it is sufficient to consider the function

$$w(x, y, t) = \max_{\eta(\cdot) \in A^t(y)} \min_{\xi(\cdot) \in A^t(y)} \min_{\tau \in [0, t]} H(\xi(\tau), \eta(\tau)) \tag{2.1}$$

where $A^t(x)$ $(A^t(y))$ is the set of solutions of system (1.1) ((1.2)), realized under all possible admissible controls $u(\cdot)$ $(v(\cdot))$ in the interval $[0, t]$ with the initial condition $x(0) = x$ $(y(0) = y)$. As is well known from [6] if conditions a)-d) are fulfilled, the sets $A^t(x)$ $(A^t(y))$ for any $x \in R^n$ $(y \in R^m)$ and $t \in R_+^1$ are compact in the metric of the space of continuous functions.

We define an operator $\Phi_- : C(D_\delta) \to C(D_\delta)$ by the following rule: for any function $w(\cdot) \in C(D_\delta)$

$$\Phi_- \circ w(x, y, t) = \max_{\tau \in [0, t]} \max_{\eta(\cdot) \in A^\tau(y)} \min_{\xi(\cdot) \in A^\tau(x)} \tag{2.2}$$
$$\min \{w(\xi(\tau), \eta(\tau), t - \tau), \min_{\theta \in [0, \tau]} H(\xi(\theta), \eta(\theta))\}$$

We can show that operator Φ_- maps space $C(D_\delta)$ into itself.

Theorem 1. Operator Φ_- maps $W_-(D_\delta)$ into itself and the inequality $\Phi_- \circ w(\cdot) \geqslant w(\cdot)$ is valid for any function $w(\cdot) \in W_-(D_\delta)$.

Proof. Let $w'(\cdot) \in W_-(D_\delta)$. We show that $\Phi_- \circ w'(\cdot) \in W_-(D_\delta)$. Taking into account the remark preceding the theorem, it is sufficient to show that the function $\Phi_- \circ w'(\cdot)$ satisfies condition 3) of Definition 1 because the validity of condition 1) of this definition is obvious for function $\Phi_- \circ w'(\cdot)$. Since $w'(\cdot) \in W_-(D_\delta)$, a strategy V' of player E exists with properties relative to $w'(\cdot)$ formulated in condition 3) of Definition 1. For definiteness let $V' = V^r = (b^r, \ldots, b^\circ)$. We shall seek a strategy $\bar{V}'$ of player E, with similar properties relative to function $\Phi_- \circ w'(\cdot)$, in the form $\bar{V}' = \bar{V}^{r+1} = (\bar{b}^{r+1}, \ldots, \bar{b}^\circ)$. We determine the mapping $\bar{b}^{r+1}$ from the condition that with the point $(x, y, t) \in D_\delta^\circ$ it associates an admissible control $v'(\tau)$, $\tau \in [0, \tau'>$ to which there corresponds a solution $\eta'(\cdot) \in A^{\tau'}(y)$ realizing jointly with τ' the maximum in the right-hand side of (2.2) wherein $w(\cdot)$ should be replaced by $w'(\cdot)$. We remark that τ' can always be chosen as positive because condition 3) of Definition 1 is valid for $w'(\cdot)$.

Further, we set $\bar{b}^i = b^i, i \in [0 : r]$ everywhere on $D_\delta°$.
We shall show that the strategy $\bar{V}'$ constructed is the one required. We fix an arbitrary strategy U of player P and an arbitrary point $(x, y, t) \in D_\delta°$. Let $\xi'(\tau) = \xi(\tau, x \mid U, \bar{V}')$, $\tau \in [0, \tau']$, where τ' is determined by mapping $\bar{b}^{r+1}$. It is easy to see that we can find a strategy U' of player P such that

$$K(x, y, t \mid U, \bar{V}') = \min \{K(\xi'(\tau'), \eta'(\tau'), t - \tau' \mid U', V'), \min_{\theta \in [0, \tau']} H(\xi'(\theta), \eta'(\theta))\}$$

Hence, taking into account the property of strategy V' and the selection of control $v'(\tau)$, $\tau \in [0, \tau' >$ by means of mapping $\bar{b}^{r+1}$, we obtain

$$K(x, y, t \mid U, \bar{V}') \geqslant \min \{w'(\xi'(\tau'), \eta'(\tau'), t - \tau'), \min_{\theta \in [0, \tau']} H(\xi'(\theta), \eta'(\theta))\} \geqslant \Phi_- \circ w'(x, y, t) \tag{2.3}$$

in addition, taking into account that $\eta'(\tau) = \eta(\tau, y, v'(\cdot)) = \eta(\tau, y \mid U, \bar{V}')$, $\tau \in [0, \tau']$, we obtain

$$\Phi_- \circ w'(\xi(\tau, x \mid U, \bar{V}'), \eta(\tau, y \mid U, \bar{V}'), t - \tau) \geqslant \min_{\xi(\cdot) \in A^{\tau'}(x)} \min \{w'(\xi(\tau'), \eta'(\tau'), t - \tau'), \min_{\theta \in [0, \tau']} H(\xi(\theta), \eta'(\theta))\} = \Phi_- \circ w'(x, y, t)$$

for all $\tau \in [0, \tau']$. Thus, condition 3) of Definition 1 is fulfilled for function $\Phi_- \circ w'(\cdot)$.

To prove the last assertion of the theorem we note that the inequality

$$\Phi_- \circ w'(x, y, t) \geqslant \min \{w'(x, y, t), H(x, y)\}$$

is valid for any point $(x, y, t) \in D_\delta$. On the other hand, allowing for the choice of strategy V' of player E, we obtain $H(x, y) \geqslant K(x, y, t \mid U, V') \geqslant w'(x, y, t)$. Consequently $\Phi_- \circ w'(x, y, t) \geqslant w'(x, y, t)$ at each point $(x, y, t) \in D_\delta$. The theorem is proved.

We select an arbitrary function $w_0(\cdot) \in W_-(D_\delta)$ and we construct successive approximations: for $n > 0$

$$\Phi_- \circ w_{n-1}(\cdot) = w_n(\cdot) \tag{2.4}$$

We also construct the sequence of strategies $\{V_n\}_{n=0}^{\infty}$ of players E : V_0 is the strategy for which, according to Definition 1, the inequality $K(x, y, t \mid U, V_0) \geqslant w_0(x, y, t)$ is valid at each point $(x, y, t) \in D_\delta°$ for any strategy U of player P; each of the strategies V_n, $n > 0$ is constructed according to strategy V_{n-1} and according to the approximation $w_{n-1}(\cdot)$ in the same way as in Theorem 1. By the construction of the sequences $\{V_n\}_{n=0}^{\infty}$ and $\{w_n(\cdot)\}_{n=0}^{\infty}$ the inequality $K(x, y, t \mid U, V_n) \geqslant w_n(x, y, t)$ is valid at each point $(x, y, t) \in D_\delta°$ for any n, for any strategy U of player P.

Theorem 2. 1. For any initial approximation from $W_-(D_\delta)$ the successive approximations (2.4) converge uniformly on D_δ to some function $w^*(\cdot)$. Function $w^*(\cdot)$ is continuous on D_δ and satisfies the equation

$$\Phi_- \circ w(\cdot) = w(\cdot) \tag{2.5}$$

2. Function $w^*(\cdot)$ is the value function of the family Γ_δ of games from initial states $(x, y, t) \in \text{int } D_\delta^\circ$ and, consequently, the limit of the successive approximations (2.4) is independent of the initial approximation from $W_-(D_\delta)$.

3. For any $\varepsilon > 0$ there exists N such that for $n \geqslant N$ strategy V_n is an E-optimal strategy for player E in each game of family Γ_δ.

Proof. The proof of the uniform convergence of sequence $\{w_n(\cdot)\}_{n=0}^{\infty}$ reduces, as a consequence of its monotonicity $w_0(\cdot) \leqslant w_1(\cdot) \leqslant, \ldots \leqslant w_n(\cdot) \leqslant, \ldots$ to establishing the uniform boundedness and equicontinuity of the set of functions $\{w_n(\cdot)\}_{n=0}^{\infty}$ The corresponding proof is quite cumbersome and therefore, it is not presented here. As a corollary of the uniform convergence we obtain that that $w^*(\cdot) \in C(D_\delta)$, addition, function $w^*(\cdot)$ satisfies Eq. (2.5) (it should be noted here that the operator $\Phi_- : C(D_\delta) \to C(D_\delta)$ is continuous).

Let us show that in each game from the initial state $(x, y, t) \in \text{int } D_\delta^\circ$ we can find, for each $\varepsilon > 0$ a pair of strategies V_ε and U_ε such that the inequalities

$$\begin{aligned} K(x, y, t \mid U, V_\varepsilon) &\geqslant w^*(x, y, t) - \varepsilon \\ K(x, y, t \mid U_\varepsilon, V) &\leqslant w^*(x, y, t) + \varepsilon \end{aligned} \tag{2.6}$$

are valid for any strategies U and V This signifies that $w^*(\cdot)$ is the value function of the game family Γ_δ. The sequence $\{w_n(\cdot)\}_{n=0}^{\infty}$ converges uniformly to $w^*(\cdot)$, therefore, an N exists such that $\nu(w^*(\cdot) - w_n(\cdot)) \leqslant \varepsilon$ for all $n_0 \geqslant N$, where ν is a norm in space $C(D_\delta)$. Let $n_0 \geqslant N$. We consider an arbitrary point $(x, y, t) \in D_\delta^\circ$, then by the choice of n_0 we obtain

$$K(x, y, t \mid U, V_{n_0}) \geqslant w_{n_0}(x, y, t) \geqslant w^*(x, y, t) - \varepsilon$$

for any strategy U of player P. Thus, if we set $V_\varepsilon = V_{n_0}$, the first of inequalities (2.6) is proved. Now we note that if the second of inequalities (2.6) has been proved, the third assertion also will have been proved. Here, however, we shall assume that $(x, y, t) \in \text{int } D_\delta^\circ$. The construction of a strategy U_ε for which the second of inequalities (2.6) is valid at a point $(x, y, t) \in \text{int } D_\delta^\circ$ does not present fundamental difficulties. Indeed, since $w^*(\cdot)$ satisfies Eq. (2.5), the corresponding strategy can be constructed by using the inequality

$$\begin{aligned} w^*(x, y, t) \geqslant \min_{\xi(\cdot) \in A^\tau(x)} \min \{ & w^*(\xi(\tau), \eta(\tau), t - \tau), \\ & \min_{\theta \in [0, \tau]} H(\xi(\theta), \eta(\theta))\} \end{aligned} \tag{2.7}$$

valid for any $(x, y, t) \in D_\delta^\circ$, $\tau \in [0, t]$ and $\eta(\cdot) \in A^\tau(y)$. The formal construction of strategy U_ε is effected with the aid of an auxiliary game in which knowledge of the preceding realization of player E control on some small interval is used and whose current state is considered to be the state $(\xi, y, t - h)$, $\xi \in C^h(x)$ for a sufficiently small $h > 0$ and not the state (x, y, t) of the original game. The latter calls for restricting the use of inequality (2.7) only for points $(\xi, y, t-h) \in D_\delta^\circ$, $\xi \in C^h(x)$ such that $(x, y, t) \in \operatorname{int} D_\delta^\circ$. We omit the details of the construction of strategy U_ε. Since the value function is unique on set $\operatorname{int} D_\delta^\circ$ and since $w^*(\cdot) \in C(D_\delta)$ and the closure of $\operatorname{int} D_\delta^\circ$ coincides with D_δ, the successive approximations (2.4) converge to one limit independently of the initial approximation chosen from $W_-(D_\delta)$. The theorem is proved.

3) Definition 2. Suppose that the function $w(\cdot)$ has been defined on D_δ. We shall say that function $w(\cdot)$ belongs to set $W_+(D_\delta)$ if:

1) $w(x, y, t) \leqslant H(x, y)$ for any point $(x, y, t) \in D_\delta^\circ$ and $w(x, y, 0) = H(x, y)$ for any point $(x, y, 0) \in D_\delta$;

2) $w(\cdot) \in C(D_\delta)$;

3) a strategy U' of player P exists such that for any strategy V of player E the inequalities $K(x, y, t \mid U', V) \leqslant w(x, y, t)$ and $w(x(\tau, x \mid U', V), y(\tau, y \mid U', V), t-\tau) \leqslant w(x, y, t)$ are valid at each point $(x, y, t) \in D_\delta^\circ$ for all $\tau \in [0, \tau']$ with some τ', $\tau' \in (0, t]$.

Note 1. A certain difference of condition 1) of this definition from the corresponding condition in Definition 1 is explained by the fact that the inequality $w(x, y, t) \leqslant H(x, y)$, $(x, y, t) \in D_\delta^\circ$ was certainly fulfilled as a consequence of condition 3) (see the proof of Theorem 1).

Lemma 2. Set $W_+(D_\delta)$ is nonempty.

To prove the lemma it is sufficient to consider the function

$$w(x, y, t) = \min_{\xi(\cdot) \in A^t(x)} \max_{\eta(\cdot) \in A^t(y)} \min_{\tau \in [0, t]} H(\xi(\tau), \eta(\tau)) \tag{3.1}$$

We define the operator $\Phi_+ : C(D_\delta) \to C(D_\delta)$ by the following rule: for any function $w(\cdot) \in C(D_\delta)$

$$\Phi_+ \circ w(x, y, t) = \min_{\tau \in [0, t]} \min_{\xi(\cdot) \in A^\tau(x)} \max_{\eta(\cdot) \in A^\tau(y)} \min\{w(\xi(\tau), \eta(\tau), t-\tau), \min_{\theta \in [0, \tau]} H(\xi(\theta), \eta(\theta))\} \tag{3.2}$$

Theorem 3. Operator Φ_+ maps $W\ (D_\delta)$ into itself and the inequality $\Phi_+ \circ w(\cdot) \leqslant w(\cdot)$ is valid for any function $w(\cdot) \in W_+(D_\delta)$

We fix an arbitrary function $w_0(\cdot) \in W_+(D_\delta)$ and we construct successive approximations by the following rule: for $n > 0$

$$\Phi_+ \circ \overline{w}_{n-1}(\cdot) = \overline{w}_n(\cdot) \tag{3.3}$$

We also construct the sequence $\{U_n\}_{n=0}^\infty$ of strategies of player P: U_0 is the strategy for which, in accord with Definition 2, the inequality $K(x, y, t \mid U_0, V) \leqslant \overline{w}_0(x, y, t)$ is valid at each point $(x, y, t) \in D_\delta^\circ$. for any strategy V of player E; each of the strategies U_n, $n > 0$ is constructed from strategy U_{n-1} and from approximation $\overline{w}_{n-1}(\cdot)$ by analogy with Sect. 2. By the construction of the sequences $\{U_n\}_{n=0}^\infty$ and $\{\overline{w}_n(\cdot)\}_{n=0}^\infty$ the inequality $K(x, y, t \mid U_n, V) \leqslant \overline{w}_n(x, y, t)$ is valid for any n at each point $(x, y, t) \in D_\delta^\circ$ for any strategy V of player E.

Theorem 4. 1. For any initial approximation from $W_+(D_\delta)$ the successive approximation (3.3) converge uniformly on D_δ to some function $w^*(\cdot)$. Function $w^*(\cdot)$ is continuous on D_δ and satisfies the equation

$$\Phi_+ \circ w(\cdot) = w(\cdot) \tag{3.4}$$

2. Function $w^*(\cdot)$ is the value function of the family Γ_δ of game from initial states $(x, y, t) \in \text{int } D_\delta{}^\circ$ and, consequently, the limit of the successive approximations (3.3) is independent of the initial approximation from $W_+(D_\delta)$.

3. For any $\varepsilon > 0$ there exists N such that for $n \geqslant N$ strategy U_n is an ε-optimal strategy of player P in each game of family Γ_δ.

Corollary (from theorem 2 and 4). The common limit $w^*(\cdot)$ of the successive approximations (2.4) and (3.3) is the value function of the family of games from initial states $(x, y, t) \in D_\delta{}^\circ$ and in each game of this family the ε-optimal strategies can be found by using the strategy sequences $\{V_n\}_{n=0}^\infty$ and $\{U_n\}_{n=0}^\infty$.

Proof. Since $w^*(\cdot)$ is the common limit of the successive approximations (2.4) and (3.3), for any $\varepsilon > 0$ we can find N such that for all $n \geqslant N$ the inequalities

$$K(x, y, t \mid U, V_n) \geqslant w_n(x, y, t) \geqslant w^*(x, y, t) - \varepsilon$$
$$K(x, y, t \mid U_n, V) \leqslant w_n(x, y, t) \leqslant w^*(x, y, t) + \varepsilon$$

are valid at each point $(x, y, t) \in D_\delta{}^\circ$ for any strategies U and V. This proves the required statement.

Note 2 If we are interested only in the solution of the game from the initial state (x_0, y_0, T), then, as we see from the corresponding constructions, to determine the game's value and the ε-optimal strategies there is no need to imbed the state (x_0, y_0, T) into set D_δ but it is sufficient to imbed it into set D, where

$$D = \bigcup_{t \in [0, T]} C^{T-t}(x_0) \times C^{T-t}(y_0) \times \{t\}$$

State (x_0, y_0, T) was imbedded into set D_δ solely for the purposes of the proof.

Note 3. Theorems 2 and 4 contain the proof of the existence of the game's value and of the ε-optimal strategies. In contrast to [7-9], here we have proposed a construction method for strategies solving, in the sense of the third assertions in Theorems 2 and 4, the pursuit game being analysed. In addition, in those papers, treating the differential game as the limiting case of an n-step game, an essential condition in the proof of the limit theorems was that the step length in the n-step game tends to zero as $n \to \infty$. Here, however, as we see, we did not assume the a priori fulfillment of any such similar conditions.

4. Theorem 5. Let a function $w^*(\cdot)$ be defined and continuous on $R^n \times R^m \times R_+{}^1$. Also let $w^*(x, y, t) \leqslant H(x, y)$ for all $(x, y, t) \in R^n \times R^m \times R_+{}^1$ and $w^*(x, y, 0) = H(x, y)$ for all $(x, y) \in R^n \times R^m$. Then the following statements are equivalent.

1. Function $w^*(\cdot)$ is the value function $R^n \times R^n \times R_+{}^1$.
2. Function $w^*(\cdot)$ satisfies the system of equations

$$\Phi_- \circ w(\cdot) = w(\cdot), \quad \Phi_+ \circ w(\cdot) = w(\cdot) \tag{4.1}$$

3. Function $w^*(\cdot)$ satisfies the equation

$$\Phi_- \circ w(\cdot) = \Phi_+ \circ w(\cdot) \tag{4.2}$$

Proof. The implication $1. \Rightarrow 2$ follows from Theorem 2 and 4. The implication $2. \Rightarrow 1$ can be proved by introducing two auxiliary multistep games progressing turn-by-turn: in one game player P takes the first turn and in the other, player E (the latter auxiliary game was already mentioned in the proof of Theorem 2). Implication $2. \Rightarrow 3$ is obvious. Let us show that $3. \Rightarrow 2$. Suppose that function $w^*(\cdot)$ satisfies Eq.(4.2). We take an arbitrary point $(x, y, t) \in R^n \times R^m \times R_+^1$, then, allowing for the inequality $w^*(x, y, t) \leqslant H(x, y)$, we obtain

$$w^*(x, y, t) = \min\{w^*(x, y, t), H(x, y)\} \leqslant \Phi_- \circ w^*(x, y, t) =$$
$$\Phi_+ \circ w^*(x, y, t) \leqslant \min\{w^*(x, y, t), H(x, y)\} = w^*(x, y, t)$$

Consequently, function $w^*(\cdot)$ satisfies system (4.1). The theorem is proved.

Note. 4. The method of successive approximations of the value function or of the payoff minimax was examined in [1, 2] for a game of encounter at a specified instant, having the general game dynamics: $x^\cdot = f(x, u, v)$. In this connection we note that if the exposition in [4, 9] is followed, then the results of the present paper can be carried over without essential changes to the differential game with game dynamics of general form (under a saddle-point condition for the small game [4]) and, respectively, with a payoff function $\min_{\tau \in [0, t]} H(x(\tau))$. We note as well that the approach proposed here to the solving of the pursuit game being considered, as well as the similar approaches in [1-3, 5] in which other formulations of the game problems of dynamics were examined, enables us to find the solution of both regular (see [4, 1]) as well as nonregular game problems.

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